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LETTER TO THE EDITOR

Non-commutative analysis, quantum group gauge transformations and gauge fields

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Abstract. In this letter a non-commutative analytic method is given, we discuss what is a q-sequence, a q-analytic function, a q-derivative, a q-differential equation and its solution, etc, respectively. By using these terms the vector fields and the local quantum group $GL_q(2)$ gauge transformation on a quantum plane can be described. From the covariance requirement under the gauge transformations we obtain the q-covariant derivatives and the $GL_q(2)$ gauge field which is a quantum analogue of the $GL(2)$ gauge field. The solutions of the $GL_q(2)$ gauge field equation are discussed, and the explicit form of the null gauge field solution is given.

In the quantum group theory, it is an interesting open problem to ask if there is a quantum analogue of gauge field theory [1]. In this respect there have been some approaches, e.g. [2–4]. However there are still some difficulties. It seems to us the crux of the problem is that even though the algebraic generators are called 'coordinates', the quantum groups are, in fact, pure algebraic structures, and the quantum differential calculus on a quantum hyperplane is a covariant algebraic system under the linear transformations of the quantum hyperplane where there are no movable coordinates, as the real or complex coordinates, as in the classical analysis, etc. Therefore, as yet we are short of a concept of 'localization', unless other real or complex coordinates are introduced into the coefficients of algebraic elements [2–4], however they already are not the coordinates of the quantum hyperplane, and will lead to other difficulties. Therefore it is hard to consider the 'fields', the 'local gauge symmetry' and the 'field equations', etc, on a quantum hyperplane. We deem that the key to construct a quantum group gauge theory is just to surmount the above difficulties, and a gauge field theory describing non-commutative fields can be given. In this letter we suggest just such a theory.

The letter is organized as follows. Firstly we discuss a non-commutative analytic method which is already different from the original non-commutative geometry [1], each generator, indeed, plays a role as a movable coordinate as in analysis. Next, by the terms of non-commutative analysis we can discuss the vector fields on a quantum plane, the local gauge transformations, the local Lie algebra, etc. From the gauge symmetry requirement the q-covariant derivatives and the quantum group gauge fields are given. In the last case, a simple case, i.e. the 'pure gauge, or flat' solution is given.

For the sake of simplicity and clarity, in this letter the deformation parameter q takes only the real number values, and we only discuss the case of the two-dimensional quantum group $GL_q(2)$. However all results concerned can be extended to the case of the complex multiparameter and higher-dimensional quantum groups.

What is a q-sequence? A q-sequence S^q is defined as a finite or infinite countable set $S^q=\{s_0, s_1, \ldots, s_\alpha, \ldots\}$, where $s_\alpha (\alpha=0,1,\ldots)$ are form elements. Among the elements of S^q there are the addition $s_\alpha+s_\beta$ and the multiplication $s_\alpha s_\beta$, its associative, however, is not commutative in general. By the generators s_α's an infinite algebraic system $\mathscr{S}^q$ on the real field R^1 is generated. Of course, the multiplication of two elements in $\mathscr{S}^q$, generally, is also non-commutative. We require that the commutation relation must depend on the parameter q, and when $q\to 1$ the multiplication must change into commutative. Since the multiplication of S^q changes into commutative as $q\to 1$, we can agree that now this S^q will be identified with some real number set $\Lambda=\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_\alpha, \ldots\}$, and we read it in $S^q\to\Lambda$ as $q\to 1$. In this letter we find that to determine a quantum group gauge field structure, in fact, is to determine the concrete structure of some q-sequence S^q and its limit Λ as $q\to 1$.

According to the symbols of Manin [1], the quantum plane $A_q^{2|0}$ is an associative algebra, its generators are x and y,

$$xy=qyx. \tag{1}$$

The quantum plane $A_q^{0|2}$ is generated by ξ and η,

$$\xi\eta=-\frac{1}{q}\eta\xi \qquad \xi^2=\eta^2=0. \tag{2}$$

A matrix

$$T_0=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_q(2) \tag{3}$$

means that the following commutation relations hold,

$$\begin{aligned} ab&=qba & ac&=qca \\ bc&=cb & bd&=qdb \\ cd&=qdc & ad-da&=(q-q^{-1})bc \end{aligned} \tag{4}$$

where a, b, c and d commute with x, y, ξ and η. Therefore T_0 represents a linear transformation of $A_q^{2|0}$ and $A_q^{0|2}$,

$$\begin{aligned} &\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (\tilde{x}, \tilde{y}) \in A_q^{2|0} \\ &\begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad (\tilde{\xi}, \tilde{\eta}) \in A_q^{0|2}. \end{aligned} \tag{5}$$

Now, we consider what is a q-analytic function on $A_q^{2|0}$. Suppose that $f(\zeta_1, \zeta_2)$ is a real analytic function of real variables ζ_1 and ζ_2 in common meaning. Therefore f can be developed into a power series

$$f=\lambda_{\alpha\beta}x^\alpha y^\beta \qquad \lambda_{\alpha\beta}\in R^1 \text{ is constant.} \tag{6}$$

Here, and in the following, we use the Einstein summation convention, i.e. the repeated Greek indices are summed over the values $0, 1, 2, \ldots$; and the repeated Latin indices are summed over the values $1, 2$. Let S^q be the q-sequence $\bar{S}^q=\{x, y; s_{\alpha\beta}\}$ $(\alpha, \beta= 0, 1, 2, \ldots; x, y\in A_q^{2|0})$ where $s_{\alpha\beta}$'s obey the following commutation relations

$$q^{-\beta\gamma}s_{\alpha\beta}s_{\gamma\delta}=q^{-\alpha\delta}s_{\gamma\delta}s_{\alpha\beta} \qquad s_{\alpha\beta}x=xs_{\alpha\beta} \qquad s_{\alpha\beta}y=ys_{\alpha\beta}. \tag{7}$$

We consider the element $f^q(x, y)$ in $\bar{\mathscr{F}}^q$,

$$f^q(x, y) = s_{\alpha\beta} x^\alpha y^\beta. \tag{8}$$

According to the above arrangement, S^q must change into a real number set as $q \to 1$. If this set is just the set $\Lambda = \{\lambda_{\alpha\beta}\}$, i.e. $s_{\alpha\beta} \to \lambda_{\alpha\beta}$, then it is called a q-analytic function on the quantum plane $A_q^{2|0}$, or a q deformation of the real analytic function f, and we read $f^q \to f$ as $q \to 1$. As for the meaning of the commutation relations (7), it is to guarantee the consistency of the definition of f. In fact, from the relation

$$x^\alpha y^\beta \cdot x^\gamma y^\delta = q^{-\beta\gamma} x^{\alpha+\gamma} y^{\beta+\delta} \tag{9}$$

we have

$$(s_{\alpha\beta} x^\alpha y^\beta)(s_{\gamma\delta} x^\gamma y^\delta) = f^q \cdot f^q = (s_{\gamma\delta} x^\gamma y^\delta)(s_{\alpha\beta} x^\alpha y^\beta). \tag{10}$$

In the following, to determine a q-analytic function f^q means that some concrete algebraic structures of S^q are given unless (7) and the limit, real analytic function f, are fixed. At present, temporarily, we don't discuss the function $f^q(\xi, \eta)$. It, in fact, has only four non-zero terms: $f^q(\xi, \eta) = s_{00} + s_{10}\xi + s_{01}\eta + s_{11}\xi\eta$, and has no real limit as $q \to 1$. In the following discussion, each q-analytic function f^q is just $f^q(x, y)$, and sometimes it is simply written as f. Notice that a real analytic function f can have the distinct q deformations, since in a S^q there may be some distinct additional algebraic structures.

Similar to [5-7], the q-derivatives of a q-analytic function f can be defined as

$$\begin{aligned} \partial_x(x, y) &= \frac{1}{x} \frac{f(q^2 x, q^2 y) - f(x, q^2 y)}{q^2 - 1} \\ \partial_y(x, y) &= \frac{1}{y} \frac{f(x, q^2 y) - f(x, y)}{q^2 - 1}. \end{aligned} \tag{11}$$

Therefore

$$\partial_x(x^\alpha y^\beta) = q^{2\beta}[\alpha] x^{\alpha-1} y^\beta \qquad \partial_y(x^\alpha y^\beta) = [\beta] x^\alpha y^{\beta-1} \tag{12}$$

where the q-integer $[\alpha] = (q^{2\alpha} - 1)/(q^2 - 1)$. When $q \to 1$, then ∂_x and ∂_y change into the ordinary partial derivatives. On the quantum plane $A_q^{2|0}$ there is the invariant differential structure [5-7] as follows. Let $\xi = \mathrm{d}x$, $\eta = \mathrm{d}y$ and $\mathrm{d} = \mathrm{d}x\, \partial_x + \mathrm{d}y\, \partial_y$, then there are the following relations

$$\begin{aligned} &\mathrm{d}(fg) = (\mathrm{d}f)g + f\,\mathrm{d}g \qquad d^2 = 0 \\ &xy = qyx \qquad x\,\mathrm{d}x = q^2\,\mathrm{d}x\,x \qquad x\,\mathrm{d}y = (q^2 - 1)\,\mathrm{d}x\,y + q\,\mathrm{d}y\,x \\ &y\,\mathrm{d}x = q\,\mathrm{d}x\,y \qquad y\,\mathrm{d}y = q^2\,\mathrm{d}y\,y. \end{aligned} \tag{13}$$

In the following we simply write $x^1 \equiv x$, $x^2 \equiv y$, $\partial_1 \equiv \partial_x$ and $\partial_z \equiv \partial_y$. For ∂ the Lebnitz rule is [5]

$$\partial_i(fg) = (\partial_i f)g + (O_i^k f)\partial_k g \qquad i = 1, 2 \tag{14}$$

where O_i^k is the operator left translating $\mathrm{d}x^i$, which is linear [5],

$$\begin{aligned} &f\,\mathrm{d}x^i = \mathrm{d}x^k (O_k^i f) \\ &O_k^i(fg) = (O_k^s f)(O_s^i g) \qquad O_k^i x^s = O_{kr}^{si} x^r \end{aligned} \tag{15}$$

O_{kr}^{si} are real numbers (see equation (13)) and the repeated Latin indices are summed over $1, 2$, when $q \to 1$, then $O_{kr}^{si} \to \delta_k^s \delta_r^i$, and $\mathrm{d}x^i$ commute with f.

In the above, it is pointed out that in the ordinary quantum group theory, we, generally, do not discuss the problem concerning to the 'non-commutative' differential equations and their solutions. However, in the present non-commutative analysis there is the problem of how to find the solutions of a q-differential equation. For example, to find a q-analytic solution f^q of q-differential equation

$$P(f^q, \partial_x f^q, \partial_y f^q, \ldots) = 0 \tag{16}$$

means that the following two steps are completed:

(i) We can substitute $f^q = s_{\alpha\beta} x^\alpha y^\beta$ and $\partial_x f^q = q^{2\beta}[\alpha] s_{\alpha\beta} x^{\alpha-1} y^\beta$, etc. into (16), and put the results in order. Let the coefficient of each monomial $x^\alpha y^\beta$ $(\alpha, \beta = 0, 1, 2, \ldots)$ be equal to zero, then some algebraic relations among $s_{\alpha\beta}$'s are obtained. This means that the q-differential equation (16), in fact, is an algebraic requirement adding on the q-sequence $S^q = \{s_{\alpha\beta}\}$.

(ii) when $q \to 1$ (16) changes into an ordinary partial differential equation. Suppose that there is a real analytic solution $f = \lambda_{\alpha\beta} x^\alpha y^\beta$, therefore let the limit of S^q be the real number set $\Lambda = \{\lambda_{\alpha\beta}\}$ as $q \to 1$. Of course, here we already suppose that the corresponding real analytic solution, indeed, exists; conversely, if there is not any real analytic solution, then the above non-commutative analytic method loses efficacy.

As an example, we consider the q-differential equation

$$(\partial_x f^q)^2 + (f^q)^2 - 1 = 0. \tag{17}$$

According to (i), we can obtain a q-analytic solution $f^q = s_{\alpha\beta} x^\alpha y^\beta$, where in the q-sequence $S^q = \{s_{\alpha\beta}\}$, besides the commutation relations given by (7), there is the following algebraic relation

$$\sum_{\substack{\alpha+\gamma=M+2\\ \beta+\delta=N}} q^{2N-\delta(\alpha-1)}[\alpha][\gamma] S_{\gamma\delta} S_{\alpha\beta} - \sum_{\substack{\alpha+\gamma=M\\ \beta+\delta=N}} q^{-\beta\gamma} S_{\alpha\beta} S_{\gamma\delta} = 0. \tag{18}$$

As for the limit f, for instance, we can take

$$f = \sin(x+y) = \sum_{\alpha,\beta=0}^{\infty} \frac{(-1)^\alpha x^{2\alpha-\beta+1} y^\beta}{\beta!(2\alpha-\beta+1)!}.$$

Therefore, when $q \to 1$,

$$s_{\alpha\beta} \to \lambda_{\alpha\beta} = \begin{cases} (-1)^\Gamma \quad [(2\Gamma+\beta+1)!\beta!]^{-1} & \text{when } \alpha+\beta = 2\Gamma-1 \\ 0, & \text{when } \alpha+\beta \neq 2\Gamma-1 \end{cases} \tag{19}$$

$$(\Gamma = 0, 1, 2, \ldots).$$

Now we enter into the discussion about quantum group gauge fields. Let $S^q_X = \{x_{\alpha\beta}\}$ and $S^q_Y = \{Y_{\alpha\beta}\}$ be two q-sequences, among which the commutation relations are

$$\begin{aligned} &q^{-\beta\gamma} X_{\alpha\beta} X_{\gamma\delta} = q^{-\alpha\delta} X_{\gamma\delta} X_{\alpha\beta} \qquad q^{-\beta\gamma} Y_{\alpha\beta} Y_{\gamma\delta} = q^{-\alpha\delta} Y_{\gamma\delta} Y_{\alpha\beta} \\ &q^{-\beta\gamma} X_{\alpha\beta} Y_{\gamma\delta} = q^{1-\alpha\delta} Y_{\gamma\delta} X_{\alpha\beta} \end{aligned} \tag{20}$$

(no summing for the repeated indices). Therefore the q-analytic functions $X(x, y) = X_{\alpha\beta} x^\alpha y^\beta$ and $Y(x, y) = Y_{\alpha\beta} x^\alpha y^\beta$ obey the commutation relation

$$XY = qYX \tag{21}$$

this means that a quantum plane $A_q^{2|0}$ is constructed again by X and Y. However, it is different from the ordinary non-commutative geometry [1] where the higher terms of x and y have to appear in X and Y. Here x and y are more like the movable coordinates of a two-dimensional analytic surface in mathematics. Thus the mapping $(x, y) \to (X, Y)$ can be explained as a 'nonlinear transformation' of the quantum plane $A_q^{2|0}$. Next we consider the q-sequence $S_T^q = \{T_{\alpha\beta}\}$ as follows:

$$T_{\alpha\beta} = \begin{pmatrix} A_{\alpha\beta} & B_{\alpha\beta} \\ C_{\alpha\beta} & D_{\alpha\beta} \end{pmatrix} \qquad (A_{00}, B_{00}, C_{00}, D_{00} \neq 0) \tag{22}$$

the commutation relations are

$$\begin{aligned} &q^{-\beta\gamma}A_{\alpha\beta}B_{\gamma\delta} = q^{1-\alpha\gamma}B_{\gamma\delta}A_{\alpha\beta} \qquad && q^{-\beta\gamma}A_{\alpha\beta}C_{\gamma\delta} = q^{1-\alpha\gamma}C_{\gamma\delta}A_{\alpha\beta} \\ &q^{-\beta\gamma}B_{\alpha\beta}C_{\gamma\delta} = q^{-\alpha\gamma}C_{\gamma\delta}B_{\alpha\beta} && q^{-\beta\gamma}B_{\alpha\beta}D_{\gamma\delta} = q^{1-\alpha\delta}D_{\gamma\delta}B_{\alpha\beta} \\ &q^{-\beta\gamma}C_{\alpha\beta}D_{\gamma\delta} = q^{1-\alpha\delta}D_{\gamma\delta}C_{\alpha\beta} \\ &q^{-\beta\gamma}A_{\alpha\beta}D_{\gamma\delta} - q^{-\alpha\delta}D_{\gamma\delta}A_{\alpha\beta} = (q-q^{-1})q^{-\beta\gamma}B_{\alpha\beta}C_{\gamma\delta} \end{aligned} \tag{23}$$

and

$$\begin{aligned} &q^{-\beta\gamma}\Psi_{\alpha\beta}\Phi_{\gamma\delta} = q^{-\alpha\delta}\Phi_{\gamma\delta}\Psi_{\alpha\beta} \\ &\Psi = A, B, C, D \qquad \Phi = X, Y \end{aligned} \tag{24}$$

where in (23) and (24) the repeated indices are not summed. Let $A = A(x, y) = A_{\alpha\beta}x^\alpha y^\beta$, and $B = B(x, y) = B_{\alpha\beta}x^\alpha y^\beta$, etc, then it is easily seen that the q-analytic functions A, B, C and D obey the same commutation relations as in (4), and (A, B, C, D) pairwise commute with (X, Y). This means that with respect to X and Y we have

$$T(x, y) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_q(2). \tag{25}$$

However, it is different from the ordinary quantum group theory where the coordinates x and y are contained in T. Therefore $T(x, y)$ is a locally linear transformation of the quantum plane $A_q^{2|0}$. In addition, it is easily verified that the 'constant part' of T is just an ordinary T_0 as in (3),

$$T_0 = \begin{pmatrix} A_{00} & B_{00} \\ C_{00} & D_{00} \end{pmatrix} \in GL_q(2). \tag{26}$$

In a gauge field the Lie algebras will be used. However the ordinary quantum groups are 'constant', their Lie algebras, as in the results of [8], are also 'constant'. Therefore, in the first place, the Lie algebras of the quantum group must be 'localized'. Let $S^q = \{(E_i)_{\alpha\beta}\}$ ($i = 1, 2$ and $\alpha, \beta = 0, 1, 2, \ldots$) be a matrix q-sequence, where each $(E_i)_{\alpha\beta}$ is a matrix

$$\begin{aligned} &(E_i)_{\alpha\beta} = \begin{pmatrix} (\omega_i^1)_{\alpha\beta} & (\omega_i^+)_{\alpha\beta} \\ (\omega_i^-)_{\alpha\beta} & (\omega_i^2)_{\alpha\beta} \end{pmatrix} \\ &(\omega_i^\phi)_{00} \neq 0 \qquad (\phi = 1, 2, +, -). \end{aligned} \tag{27}$$

The commutation relations, in which the translation operator O_k^i is used, between the q-sequence S_E^q and the q-sequence S_T^q are taken as

$$\begin{aligned} &(O_i^k A_{\alpha\beta})(\omega_k^\pm)_{\gamma\delta} = q^{1+\beta\gamma-\alpha\delta}(\omega_i^\pm)_{\gamma\delta}A_{\alpha\beta} \\ &(O_i^k C_{\alpha\beta})(\omega_k^\pm)_{\gamma\delta} = q^{-1+\beta\gamma-\alpha\delta}(\omega_i^\pm)_{\gamma\delta}C_{\alpha\beta} \\ &(O_i^k A_{\alpha\beta})(\omega_k^1)_{\gamma\delta} = q^{2+\beta\gamma-\alpha\delta}(\omega_i^1)_{\gamma\delta}A_{\alpha\beta} \\ &(O_i^k A_{\alpha\beta})(\omega_k^2)_{\gamma\delta} = q^{\beta\gamma-\alpha\delta}(\omega_i^2)_{\gamma\delta}A_{\alpha\beta} \\ &(O_i^k C_{\alpha\beta})(\omega_k^1)_{\gamma\delta} = q^{-2+\beta\gamma-\alpha\delta}(\omega_i^1)_{\gamma\delta}C_{\alpha\beta} \\ &(O_i^k C_{\alpha\beta})(\omega_k^2)_{\gamma\delta} = q^{\beta\gamma-\alpha\beta}(\omega_i^2)_{\gamma\delta}C_{\alpha\beta} \end{aligned} \tag{28}$$

and again $(A, C) \to (B, D)$ gives the remaining relations. In addition, there are the following commutation relations in E_i

$$
\begin{aligned}
&q^{-\beta\gamma} O_j^k(\omega_i^1)_{\alpha\beta} \cdot (\omega_k^2)_{\gamma\delta} + q^{-\alpha\delta} O_j^k(\omega_i^2)_{\gamma\delta} \cdot (\omega_k^1)_{\alpha\beta} = 0 \\
&q^{-\beta\gamma} O_j^k(\omega_i^+)_{\alpha\beta} \cdot (\omega_k^-)_{\gamma\delta} + q^{2-\alpha\delta} O_j^k(\omega_i^-)_{\gamma\delta} \cdot (\omega_k^+)_{\alpha\beta} = 0 \\
&q^{-\beta\gamma} O_j^k(\omega_i^\pm)_{\alpha\beta} \cdot (\omega_k^1)_{\gamma\delta} + q^{\pm 2-\alpha\delta} o_j^k(\omega_i^1)_{\gamma\delta} \cdot (\omega_k^\pm)_{\alpha\beta} = 0 \\
&q^{-\beta\gamma} O_j^k(\omega_i^\pm)_{\alpha\beta} \cdot (\omega_k^2)_{\gamma\delta} + q^{-\alpha\delta} O_j^k(\omega_i^2)_{\gamma\delta} \cdot (\omega_k^\pm)_{\alpha\beta} = 0.
\end{aligned}
\tag{29}
$$

In fact, the above relations in (28) and (29) guarantee that just the following localized commutation relations hold:

$$
\begin{aligned}
&\Omega^1\Omega^2 + \Omega^2\Omega^1 = 0 \qquad && \Omega^+\Omega^- + q^2\Omega^-\Omega^+ = 0 \\
&\Omega^\pm\Omega^1 + q^{\pm 2}\Omega^1\Omega^\pm = 0 \qquad && \Omega^\pm\Omega^2 + \Omega^2\Omega^\pm = 0
\end{aligned}
\tag{30}
$$

where $\Omega^\phi = \mathrm{d}x\, \omega_1^\phi + \mathrm{d}y\, \omega_2^\phi$, $\omega_i^\phi = (\omega_i^\phi)_{\alpha\beta} x^\alpha y^\beta$, $(\phi = 1, 2, +, -$ and $i = 1, 2)$, and

$$
\begin{aligned}
&A\Omega^1 = q^2\Omega^1 A \qquad && A\Omega^2 = \Omega^2 A \\
&C\Omega^1 = \frac{1}{q^2}\Omega^1 C \qquad && C\Omega^2 = \Omega^2 C.
\end{aligned}
\tag{31}
$$

The remaining relations are given by $(A, C) \to (B, D)$ again. Equations (30) and (31) mean that the matrix

$$
E = \mathrm{d}x^i E_i = \mathrm{d}x^i \begin{pmatrix} \omega_i^1 & \omega_i^+ \\ \omega_i^- & \omega_i^2 \end{pmatrix} = \begin{pmatrix} \Omega^1 & \Omega^+ \\ \Omega^- & \Omega^2 \end{pmatrix}
\tag{32}
$$

belongs to the Lie algebra of $GL_q(2)$ [8]. However, since ω_i^ϕ contains the coordinates x and y, $E_i(x, y)$ is localized. If $(\omega_i^\phi)_{\alpha\beta} = 0$ for $\alpha, \beta \geqslant 1$, then they return to the results in [8].

Now according to (21) and (24), the q-analytic function pair $V = V(x, y) = (X(x, y), Y(x, y))$ can be explained as a q-vector field on the quantum plane $A_q^{2|0}$, and $T(x, y)$ represents a local $GL_q(2)$ gauge transformation

$$
V \to \tilde{V} = TV.
\tag{33}
$$

In this letter we want to construct a q deformation field theory with this local gauge symmetry. For this purpose, we define the q-covariant differential calculus $\mathscr{D}$ by

$$
\mathscr{D} = d + E
\tag{34}
$$

where E is the Lie algebra of quantum group $GL_q(2)$ described as in (27)-(32). The covariance means the requirement

$$
\tilde{\mathscr{D}}(TV) = T(\mathscr{D}V) \qquad \text{i.e. } (d + E)(TV) = T(d + E)V.
\tag{35}
$$

From this we obtain that the gauge transformation of E is

$$
E \to \tilde{E} = TET^{-1} - (\mathrm{d}T)T^{-1}.
\tag{36}
$$

where $E = \mathrm{d}x^i E_i$, E_i $(i = 1, 2)$ is a q-analytic function matrix as in (32), it is the $GL_q(2)$ gauge field (or gauge potential). Therefore we can write

$$
\mathscr{D} = \mathrm{d}x^i\, \nabla_i \qquad \nabla_i = \partial_i + E_i.
\tag{37}
$$

∇_i can be called the q-covariant derivative. However, the meaning of the covariance is

$$
\tilde{\nabla}_i(TV) = (O_i^k T)\nabla_k V
\tag{38}
$$

where O_i^k is the left translation operator defined by (15). Therefore the gauge transformation of ∇ is

$$\nabla_i \to \tilde{\nabla}_i (O_i^k T) \nabla_k T^{-1}. \tag{39}$$

As for the gauge field intensity F_{ij}, we find that it may be constructed as follows. In the first place, let $\mathscr{R}_i^k$ be the operator right translating $\mathrm{d}x^i$, which is defined by

$$\mathrm{d}x^i\, \nabla_k = (\mathscr{R}^i_m \nabla_k)\, \mathrm{d}x^m \tag{40}$$

and

$$\mathscr{R}^i_m \nabla_k = \mathscr{R}^i_m(\partial_k + E_k) = P^i_m \partial_k + Q^i_m E_k \tag{41}$$

where Q^i_m is the operator right translating $\mathrm{d}x^i$ through a q-analytic function by using (13), and P^i_m is the operator right translating $\mathrm{d}x^i$ through ∂, of which the concrete expressions are given in [5]. Obviously, $Q = O^{-1}$ and we have

$$\begin{aligned} &\mathrm{d}x^i f = (Q^i_k f)\, \mathrm{d}x^k \\ &Q^i_k x^s = Q^{si}_{kr} x^r \\ &Q^i_m(fg) = (Q^k_m f)(Q^i_k g) \\ &Q^i_m O^m_j = O^m_j Q^i_m = \delta^i_j \cdot 1 \end{aligned} \tag{42}$$

where Q^{si}_{kr} are real numbers which are determined by (13), and 1 is the identity operator. By using the above symbols, the $GL_q(2)$ gauge field intensity F_{ij} is defined as

$$F_{ij} = \partial_i (Q^k_j E_k) - \partial_j (Q^k_i E_k) + [E_i, E_j]_Q \tag{43}$$

where $[\,,]_Q$ is a deformation Lie bracket

$$[E_i, E_j]_Q = E_i(Q^k_j E_k) - E_j(Q^k_i E_k). \tag{44}$$

F_{ij} is an anti-symmetric, $F_{ij} = -F_{ji}$, it has only one essential component, i.e. F_{12} or F_{21}. By using (40)–(44) and a direct calculation, it can be proved that the gauge transformation of F_{ij} is

$$F_{ij} \to \tilde{F}_{ij} = (O^k_i T) F_{km} (Q^m_j T^{-1}). \tag{45}$$

However, the simplest way is that F_{ij} can be written as

$$F_{ij} = \nabla_i (Q^k_j E_k) - \nabla_j (Q^k_i E_k) \tag{46}$$

then the correctness of (45) is easily seen from (36), (39) and (42). Since $Q^k_i \to \delta^k_i \cdot 1$ as $q \to 1$, F_{ij} changes into the ordinary gauge field intensity as $q \to 1$.

F_{ij} describes a non-commutative quantum deformation field. Now, we consider the source-free field equation and its solutions. Equations

$$\nabla_i F_{ij} = 0 \qquad j = 1, 2 \tag{47}$$

in fact, contain only two non-zero equations, i.e.

$$\partial_1 F_{12} = E_1 F_{12} \qquad \partial_2 F_{12} = E_2 F_{12}. \tag{48}$$

In order to obtain the solutions we, in the first place, must calculate the concrete results of the operator Q. From (13) and $E_i = (E_i)_{\alpha\beta} x^\alpha y^\beta$, we have

$$\begin{aligned} &\mathrm{d}x\, E_i = (E_i)_{\alpha\beta} q^{-2\alpha-\beta} x^\alpha y^\beta\, \mathrm{d}x \\ &\mathrm{d}y\, E_i = (E_i)_{\alpha\beta} [q^{1-3\alpha-\beta}(1-q^{2\alpha}) x^{\alpha-1} y^{\beta+1}\, \mathrm{d}x + q^{-\alpha-2\beta} x^\alpha y^\beta\, \mathrm{d}y] \end{aligned} \tag{49}$$

therefore

$$Q_1^k E_k = (E_1)_{\alpha\beta} q^{-2\alpha-\beta} x^\alpha y^\beta + (E_2)_{\alpha\beta} q^{1-3\alpha-\beta}(1-q^{2\alpha}) x^{\alpha-1} y^{\beta+1}$$
$$Q_2^k E_k = (E_2)_{\alpha\beta} q^{-\alpha-2\beta} x^\alpha y^\beta. \tag{50}$$

We substitute (50) into (43), (44) and (48), and after through a careful calculation we can obtain the concrete algebraic relations in the q-sequence $S^q = \{(E_i)_{\alpha\beta}\}$, however they are more lengthy in form. As an example, here we write explicitly its null gauge field solution as follows. From the null field

$$F_{12} = 0$$

i.e.

$$\begin{aligned} q^{-\alpha}[\alpha](E_2)_{\alpha\beta} x^{\alpha-1} y^\beta - q^{-2\alpha-\beta}[\beta](E_1)_{\alpha\beta} x^\alpha y^{\beta-1} \\ - q^{1-3\alpha-\beta}(1-q^{2\alpha})[\beta+1](E_2)_{\alpha\beta} x^{\alpha-1} y^\beta + q^{-\alpha-2\beta-\beta\gamma}(E_1)_{\alpha\beta} x^{\alpha+\gamma} y^{\beta+\delta} \\ - q^{-2\gamma-\delta-\beta\gamma}(E_2)_{\alpha\beta}(E_1)_{\gamma\delta} x^{\alpha+\gamma} y^{\beta+\delta} \\ + q^{1-3\gamma-\beta\gamma+\beta\delta}(1-q^{2\gamma})(E_2)_{\alpha\beta}(E_1)_{\gamma\delta} x^{\alpha+\gamma-1} y^{\beta+\delta-1} = 0 \end{aligned} \tag{51}$$

we obtain the algebraic relation in S^q

$$\begin{aligned} \sum_{\substack{\alpha+\gamma=M \\ \beta+\delta=N}} q^{-\gamma-2\delta-\beta\gamma}(E_1)_{\alpha\beta}(E_2)_{\gamma\delta} \\ + \sum_{\substack{\alpha+\gamma=M+1 \\ \beta+\delta=N+1}} q^{1+\beta+3\gamma+\delta-\beta\gamma}(1-q^{-2\gamma})(E_1)_{\alpha\beta}(E_2)_{\gamma\delta} \\ = q^{-1-2M-N}[N+1](E_1)_{M,N+1} \\ -(q^{-M-1}[M+1] + q^{-2-3M-N}(1-q^{2(M+1)})[N+1])(E_2)_{M+1,N} \end{aligned} \tag{52}$$

where the integers M, $N = 0, 1, 2, \ldots$. As for the limit as $q \to 1$, we only need to take ω_i^ϕ ($i = 1, 2$ and $\phi = 1, 2, +, -$) as the ordinary real functions, and to seek a real analytic solution $\Lambda_i = [\lambda_i^\phi]$ of the following common first-order partial differential equation (such Λ_i exists)

$$\frac{\partial}{\partial x}\Lambda_2 - \frac{\partial}{\partial y}\Lambda_1 + \Lambda_1\Lambda_2 - \Lambda_2\Lambda_1 = 0. \tag{53}$$

Therefore we make $E_i \to \Lambda_i$ or $\omega_i^\phi \to \lambda_i^\phi$ as $q \to 1$.

In summary, in order to describe some nonlinear and non-commutative fields, a quantum group gauge field theory is necessary and possible. In this letter the above discussion is just such a scheme. In addition, by use of the Yang–Baxter matrix terms the above results can be extended to the case of complex multiparameter and higher-dimensional quantum group; also to the direct product quantum groups [9], etc. These and some possible applications will be discussed by us elsewhere.

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